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# Sequential Compactness in Constructive Analysis 

By

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#### Abstract

A new constructive notion of sequential compactness is introduced, and its relation to completeness and totally boundedness is explored.


In this note we complement the work in [3] by introducing, within the framework of (Bishop's) constructive mathematics [1], a new approach to sequential compactness. We begin with the fundamental definition on which the paper is based.

A sequence $\mathbf{x}=\left(x_{n}\right)$ in a metric space $(X, \rho)$ has at most one cluster point if the following condition holds:

There exists $\delta_{\mathbf{x}}>0$ such that if $0<\delta<\delta_{\mathbf{x}}$ and $\rho(a, b)>2 \delta$, then either $\rho\left(x_{n}, a\right)>\delta$ for all sufficiently large $n$ or else $\rho\left(x_{\mu}, b\right)>\delta$ for
all sufficiently large $n$.
Note that each subsequence of $\left(x_{n}\right)$ then has at most one cluster point: indeed, the same $\delta_{\mathbf{x}}$ works for such a subsequence as for the original sequence $\mathbf{x}$.

A Cauchy sequence $\mathbf{x}$ has at most one cluster point. To see this, let $\rho(a, b)>2 \delta>0$. Choose $\varepsilon>0$ such that $\rho(a, b)>2(\delta+\varepsilon)$, and then choosen $N$ such that $\rho\left(x_{m}, x_{n}\right)<\varepsilon$ for all $m, n \geq N$. Since

$$
\left(\rho\left(x_{N}, a\right)-\delta-\varepsilon\right)+\left(\rho\left(x_{N}, b\right)-\delta-\varepsilon\right) \geq \rho(a, b)-2(\delta+\varepsilon)>0
$$

either $\rho\left(x_{N}, a\right)>\delta+\varepsilon$ or $\rho\left(x_{N}, b\right)>\delta+\varepsilon$. Inthefirstcase, $\rho\left(x_{n}, a\right)>\delta$ for all $n \geq N$; in the second, $\rho\left(x_{n}, b\right)>\delta$ for all $n \geq N$.

We call $X$ sequentially compact if every sequence in $X$ that has at most one cluster point converges to a limit in $X$. To see that this notion of sequential compactness is classically equivalent to the usual one, ${ }^{1}$ suppose that $X$ is sequentially compact in our sense, and let $\left(x_{n}\right)$ be any sequence in $X$; if $\left(x_{n}\right)$ does not have a cluster point, then it has at most one cluster point and so converges in $X$, a contradiction. On the other hand, suppose that $X$ is sequentially compact in the usual sense, and consider a sequence $\left(x_{n}\right)$ in $X$ that has at most one cluster point. Since $X$ is classically sequentially compact, there exists a subsequence $\left(x_{n_{k}}\right)_{k=1}^{\infty}$ of $\left(x_{n}\right)$ that converges to a limit $x_{\infty}$ in $X$. If $\left(x_{n}\right)$ does not converge to $x_{\infty}$, then there exists a subsequence of $\left(x_{n}\right)$ that is bounded away from $x_{\infty}$; this subsequence has cluster points, but none of those can equal $x_{\infty}$; this contradicts our hypothesis that $\left(x_{n}\right)$ has at most one cluster point.

Classically, a metric space is sequentially compact if and only if it is complete and totally bounded ([4], (3.16.1)). There is a natural approximate interval-halving proof that $[0,1]$ is constructively sequentially compact in our sense. Given a sequence $\left(x_{n}\right)$ in $[0,1]$ that has at most one cluster point, let $I_{0}=[0,1]$. Taking $a=\frac{1}{5}$ and $b=\frac{4}{5}$ in the definition of at most one cluster point, we see that as $|a-b|>\frac{2}{5}$,
$\triangleright$ either $\left|x_{n}-\frac{1}{5}\right|>\frac{1}{5}$, and therefore $x_{n}>\frac{2}{5}$, for all sufficiently large $n$;
$\triangleright$ or else $\left|x_{n}-\frac{4}{5}\right|>\frac{1}{5}$, and therefore $x_{n}<\frac{3}{5}$, for all sufficiently large $n$.
In the first case, take $I_{1}=\left[\frac{2}{5}, 1\right]$; in the second, take $I_{1}=\left[0, \frac{3}{5}\right]$. Carrying on in this way, we produce closed intervals $I_{0} \supset I_{1} \supset I_{2} \supset$ such that for each $n, \left.\left|I_{n}\right|=\frac{3}{5} \right\rvert\, I_{n-1} V$ and $x_{k} \in I_{n}$ for all sufficiently large $k$. Then there exists a unique point $x_{\infty} \in \bigcap_{n=0}^{\infty} I_{n}$, and it is routine to show that $x_{\infty}=\lim _{n \rightarrow \infty} x_{n}$.

The following key lemma will enable us to generalise this from $[0,1]$ to any complete, totally bounded metric space.

Lemma 1. Let $\mathbf{x}=\left(x_{n}\right)$ be a sequence with at most one cluster point in a metric space $X$, let $\delta_{\mathbf{x}}$ be as in the foregoing definition, and let $0<\varepsilon<\delta_{\mathbf{x}}$. Suppose that there exists a finitely enumerable ${ }^{2}$ subset $F$ of $X$ such that for each n there exists $x \in F$ with $\rho\left(x, x_{n}\right)<\varepsilon$. Then $\rho\left(x_{m}, x_{n}\right)<8 \varepsilon$ for all sufficiently large $m$ and $n$.

[^0]Proof: Let $\xi_{1} \in F$. Either $\rho\left(\xi, \xi_{1}\right)<3 \varepsilon$ for all $\xi \in F$ or else there exists $\xi^{\prime} \in F$ such that $\rho\left(\xi^{\prime}, \xi_{1}\right)>2 \varepsilon$. In the first case we have $\rho\left(x_{n}, \xi_{1}\right)<4 \varepsilon$ for all $n$, and therefore $\rho\left(x_{m}, x_{n}\right)<8 \varepsilon$ for all $m$ and $n$; so we may assume that the second case obtains. Accordingly, by our hypothesis on $\mathbf{x}$, either $\rho\left(x_{n}, \xi_{1}\right)>\varepsilon$ for all sufficiently large $n$ or else $\rho\left(x_{n}, \xi^{\prime}\right)>\varepsilon$ for all sufficiently large $n$. Interchanging $\xi_{1}$ and $\xi^{\prime}$, if necessary, we may assume that $\rho\left(x_{n}, \xi_{1}\right)>\varepsilon$ for all $n \geq N_{1}$. If follows that for each $n \geq N_{1}$ there exists

$$
\xi \in F \sim\left\{\xi_{1}\right\}=\left\{x \in F \quad x \neq \xi_{1}\right\}
$$

such that $\rho\left(x_{\mu \prime}, \xi\right)<\varepsilon$. We may therefore repeat the foregoing argument, with $\mathbf{x}$ replaced by $\left(x_{n}\right)_{n \geq N_{1}}$ and $F$ replaced by $F \sim\left\{\xi_{1}\right\}$. In this way we obtain $\xi_{2} \in F \sim\left\{\xi_{1}\right\}$ such that
$\triangleright$ either $\rho\left(x_{n}, \xi_{2}\right)<4 \varepsilon$ for all $n \geq N_{1}$, and therefore $\rho\left(x_{m}, x_{n}\right)<8 \varepsilon$ for all $m, n \geq N_{1}$,
$\triangleright$ or else there exists a positive integer $N_{2}>N_{1}$ such that $\rho\left(x_{n}, \xi_{2}\right)>\varepsilon$ for all $n \geq N_{2}$.
Executing this procedure a total of at most \#F times, we are guaranteed to produce $N$ such that $\rho\left(x_{m}, x_{n}\right)<8 \varepsilon$ for all $m, n \geq N$.Q.E.D.

Corollary 2. If $X$ is a totally bounded metric space, then any sequence in $X$ with at most one cluster point is a Cauchy sequence.

Corollary 3. The following are equivalent conditions on a sequence $\left(x_{n}\right)$ in any metric space $X$ :
(i) $\left(x_{n}\right)$ is totally bounded and bas at most one cluster point.
(ii) $\left(x_{n}\right)$ is a Caucby sequence.

The following constructive generalisation of the Bolzano-Weierstraß Theorem is an immediate consequence of Corollary 2.

Theorem 4. A complete, totally bounded metric space is sequentially compact.
We now prove some partial converses of this theorem.
Proposition 5. If $X$ is sequentially compact, then it is complete.
Proof: Every Cauchy sequence in $X$ has at most one cluster point and so converges. Q.E.D.

Proposition 6. Let $X$ be sequentially compact, and let a be a point of $X$ such that for all positives, $t$ with $s<t$, either $\rho(x, a)<t$ for all $x \in X$ or else $\rho(x, a)>s$ for some $x \in X$. Then $X$ is bounded.

Proof: Construct an increasing binary sequence $\left(\lambda_{n}\right)$ such that
$\triangleright$ if $\lambda_{n}=0$, then there exists $x \in X$ such that $\rho(x, a)>n$,
$\triangleright$ if $\lambda_{n}=1$, then $\rho(x, a)<n+1$ for all $x \in X$.
We may assume that $\lambda_{1}=0$. If $\lambda_{n}=0$, choose $x_{n} \in X$ such that $\rho\left(x_{n}, a\right)>n$; if $\lambda_{n}=1$, set $x_{n}=x_{n-1}$. To prove that $\mathbf{x}=\left(x_{n}\right)$ has at most one cluster point, let $\rho(y, z)>2 \delta>0$, and choose a positive integer

$$
N>\max \left\{\rho\left(a, y^{\prime}\right), \rho(a, z)\right\}+\delta
$$

If $\lambda_{N}=1$, then $x_{n}=x_{N}$ for each $n \geq N$, so that either $\rho\left(x_{n}, y\right)>\delta$ for all $n \geq N$ or else $\rho\left(x_{n}, z\right)>\delta$ for all $n \geq N$. Consider, on the other hand, what happens if $\lambda_{N}=0$. If $n \geq N$ and $\lambda_{n}=0$, then $\rho\left(x_{n}, a\right)>n \geq N$, so

$$
\rho\left(x_{n}, y^{\prime}\right) \geq \rho\left(x_{n}, a\right)-\rho\left(a, y^{\prime}\right)>\delta
$$

and likewise $\rho\left(x_{n}, z\right)>\delta$. If $n \geq N$ and $\lambda_{n}=1$, then there exists $k \in\{N+1, \quad, n\}$ such that $\lambda_{k}=1-\lambda_{k-1}$; whence $x_{n}=x_{n-1}=$ $=x_{k-1}$ where, as above, $\rho\left(x_{k-1}, y\right)>\delta$ and $\rho\left(x_{k-1}, z\right)>\delta$.

Thus $\mathbf{x}$ has at most one cluster point in $X$ and therefore converges to a limit $x_{\infty} \in X$. Choosing a positive integer $n>1+\rho\left(x_{\infty}, a\right)$ such that $\rho\left(x_{n}, x_{\infty}\right)<1$, we see that $\lambda_{n}=1$. Q.E.D.

The constructive least-upper-bound principle states that if the nonempty subset $S$ of $\mathbf{R}$ is not only bounded above, but also located - in the sense that for all $\alpha, \beta$ with $\alpha<\beta$, either $\beta$ is an upper bound of $S$ or else there exists $x \in S$ with $x>\alpha$-then $\sup S$ exists. The locatedness condition cannot be dropped constructively, although it is redundant classically.

Corollary 7. Under the hypotheses of Proposition 6, $\sup _{x \in X} \rho(x, a)$ exists.
Proof: Since $X$ is bounded by Proposition 6, we can apply the least-upperbound principle to the set $\{\rho(x, a) \quad x \in X\}$. Q.E.D.

Proposition 8. Let X be separable and sequentially compact. Then the following conditions are equivalent.
(i) For each $\xi \in X, \sup _{x \in X} \rho(x, \xi)$ exists.
(ii) $X$ is totally bounded.

Proof: Let $\left(a_{n}\right)_{n=1}^{\infty}$ be a dense sequence in $X$, and let $\varepsilon>0$. Set $n_{0}=1$, assume (i), and construct an increasing binary sequence $\left(\lambda_{k}\right)_{k=1}^{\infty}$, and an increasing sequence $\left(n_{k}\right)_{k=1}^{\infty}$ of positive integers, such that
$\Delta$ if $\lambda_{k}=0$, then $\rho\left(a_{n_{k}},\left\{a_{1}, a_{2}, . ., a_{n_{k-1}}\right\}\right)>\varepsilon$,
$\triangleright$ if $\lambda_{k}=1$, then $\sup _{x \in X} \rho\left(x,\left\{a_{1}, a_{2}, . \quad, a_{n_{k-1}}\right\}\right)<2 \varepsilon$.

If $\lambda_{k}=0$, put $x_{k}=a_{n_{k}}$; if $\lambda_{k}=1$, put $x_{k}=x_{k-1}$. We show that the sequence $\mathbf{x}=\left(x_{k}\right)_{k=1}^{\infty}$ has at most one cluster point in $X$. To this end, let $0<\delta<\varepsilon$ and $\rho(y, z)>2 \delta$, and choose $j$ such that $\rho\left(y, a_{j}\right)<\varepsilon-\delta$. Either $\lambda_{k}=1$ for some $k \leq j$, or else $\lambda_{j}=0$. In the first case the sequence $\mathbf{x}$ is eventually constant and so clearly has at most one cluster point. In the second we may assume that $\lambda_{j+1}=0$; so if $i \geq j+1$ and $\lambda_{i}=0$, then

$$
\rho\left(y, x_{i}\right)=\rho\left(y, a_{n_{i}}\right) \geq \rho\left(a_{j}, a_{n_{i}}\right)-\rho\left(y, a_{j}\right)>\varepsilon-(\varepsilon-\delta)=\delta .
$$

It follows that if $i>j+1$ and $\lambda_{i}=1$, then, as $x_{i}=x_{k}$ for some $k \in\{j+1, \ldots, i-1\}$ with $\lambda_{k}=0$, we also have $\rho\left(y, x_{i}\right)>\delta$. This completes the proof that $\mathbf{x}$ has at most one cluster point.

Since $X$ is sequentially compact, $\mathbf{x}$ converges to a limit $x_{\infty} \in X$. Choose $\kappa$ such that $\rho\left(x_{\infty}, x_{k}\right)<\varepsilon / 2$ for all $k \geq \kappa$. Then either $\lambda_{\kappa}=1$ or else $\lambda_{\kappa}=0$; in the latter case, as $\rho\left(x_{\kappa+1}, x_{\kappa}\right)<\varepsilon$, we must have $\lambda_{\kappa+1}=1$. Hence $\left\{a_{1}, a_{2}, \ldots, a_{n_{k+1}}\right\}$ is an $\varepsilon$-approximation to $X$. This completes the proof that (i) implies (ii).

If, conversely, (ii) holds, then the uniform continuity of the mapping $x \mapsto \rho(x, \xi)$ ensures that $\sup _{x \in X} \rho(x, \xi)$ exists ([1], page 94, (4.3)). Q.E.D.

It it tempting to try working with a simpler notion of " $\mathbf{x}$ has at most one cluster point": namely, that if $a, b$ are distinct points of $X$, then either $\mathbf{x}$ is eventually bounded away from $a$, or $\mathbf{x}$ is eventually bounded away from $b$. However, Specker'sTheorem ([5]; see also [2], page 58) shows that in the recursive model of constructive mathematics there exists a sequence in $[0,1]$ which is eventually bounded away from any given recursive real number and, a fortiori, cannot converge.

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[^0]:    ${ }^{1}$ The classical property of sequential compactness does not hold constructively even for the pair set $\{0,1\}$, and so is constructively useless.
    ${ }^{2} \mathrm{~A}$ set is finitely enumerable if it is the range of a mapping $f$ from $\{1, ., n\}$, for some natural number $n$. If also $f$ is one - one, then its range is said to be finite.

